Math Teachers' Circle Lesson: How do you connect the dots?

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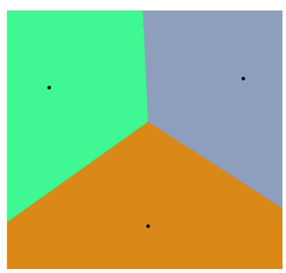
Abstract. Suppose you have a collection of dots on a page and want to find a way to connect them "automatically" in some meaningful way. How would you do it? This question has inspired mathematicians, computer scientists, statisticians, network engineers, and even artists and architects! We will learn about the exciting mathematics behind this problem primarily through drawing.

Preparation. Before teachers arrive, draw dots on boards around the room according to Voronoi examples below. Draw the dots only.

Introduction - Discussion. Draw two dots on a central board. Explain that each dot represents the house of a rancher. The ranchers want to agree on what parts of the land belong to each of them by building a fence. Draw the midline between the points and discuss this as a fair way of dividing the land.



Now introduce a third point so that the three points form a roughly equilateral triangle. Discuss how to re-draw fences now in a "fair" way - there should be three rays, emanating from a single point near the middle of the triangle formed by the points, as shown above. Discuss how each portion of fence divides space evenly between two "neighbors." Discuss how the point where the fences meet represents the point that is equidistant to all three houses.

Activity 1 - Drawing Voronoi Diagrams. Assign groups of teachers to each of the boards around the room (which already have dots pre-drawn). Ask them to draw fences that "divides the land fairly." You can either let them discover the definition or state it yourself: a fence should be placed as part of a midline between two ranches and extended until it meets another fence or goes off the board. The case of three ranches placed at the corners of a very obtuse triangle can be very illustrative

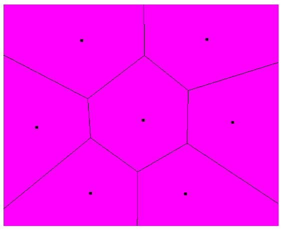
Go around to groups and assist as needed. If a group finishes early, give them a challenge question such as

- Can you place houses so that some rancher has a triangle / square / pentagon as his plot of land?

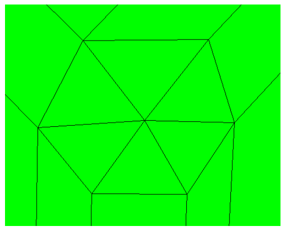
- Can you place houses so that some rancher has a circle as his plot of land?
- Draw four houses at the corners of a square. What do the plots of land look like? What happens if you move one corner 'out' a little bit? What happens if you it 'in' a little bit?

After all groups have finished their diagrams, they should go around and look at each others' pictures and reconcile any disagreements.

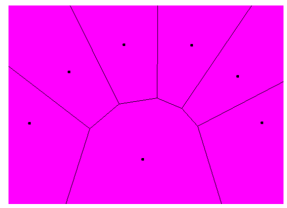
Examples for Groups Here are 4 Voronoi digaram / Delaunay triangulation pairs:



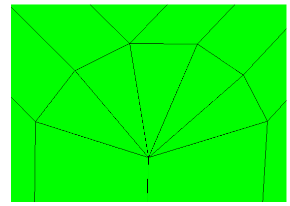
Voronoi 1



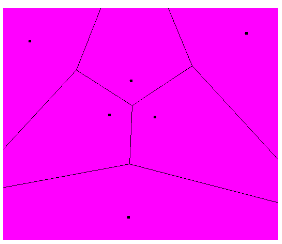
Delaunay 1



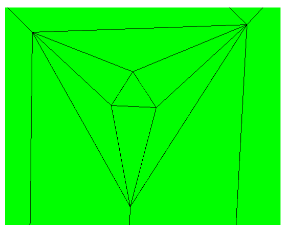
Voronoi 2



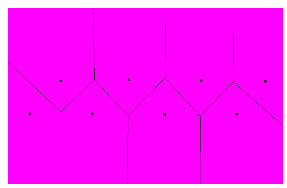
Delaunay 2



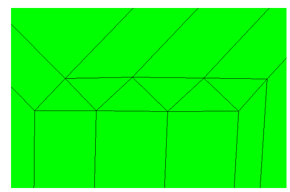
Voronoi 3



Delaunay 3



Voronoi 4



Delaunay 4

***** Break *****
Teaser/preview question: How do these diagrams relate to connecting the dots?

Activity 2: Drawing Delaunay Triangulations Explain that now the ranchers want to build tunnels to their neighbors so they can visit each other without having to go outside. They only want to build straight tunnels and only to their nearest neighbors. More precisely, define that a tunnel should be placed as a straight line between two ranches that have a fence appearing the in the fence diagram. Discuss how to do this in the easy case of 3 points in an equilateral triangle. Also do an example with an obtuse triangle and discuss what happens in the case of the "long edge" of the triangle (it's a tunnel!).

Ask groups to return to their diagrams and draw tunnels according to this process. The teachers should NOT erase the fence (Voronoi) diagram but instead carefully draw on top of it, using a different color of pen / chalk. As groups finish, ask challenge questions such as:

- Can two tunnels ever intersect, other than at a house?
- What does it mean if a house has many tunnels coming to it?
- Can you place houses so that the tunnels enclose some region other than a triangle?
- Draw four houses at the corners of a square. What do the tunnels look like? What happens if you move one corner 'out' a little bit? What happens if you it 'in' a little bit?

After all groups have finished their diagrams, they should go around and look at each others and reconcile any disagreements. Also, everyone should discuss the relation between the fence diagrams and the tunnel diagrams.

Lead a group discussion about **duality**: a tunnel will always correspond to a fence and a fence will always correspond to a tunnel. Consider the intersection of a fence-tunnel dual pair: their intersection will always be at a right angle (even if you have to extend the fence to find the point of intersection). Have groups discuss why this is and have some volunteers describe to the whole group.

Exhibition of Online Diagram Generators. Take a brief break to show some online Voronoi diagram / Delaunay triangulation generators. This one allows you to input points and then let them move randomly while it dynamically updates the Voronoi diagram:

`http://alexbeutel.com/webgl/voronoi.html`

This one lets you toggle between the Voronoi and Delaunay diagrams (you may need to update Java and change plug-in permissions to get it to work):

`http://www.cs.cornell.edu/info/people/chew/delaunay.html`

You can find others by searching for "Voronoi diagram applet" or similar terms.

Activity 3: The Empty Ball Property. Recall that any three points uniquely define a circle (unless they are collinear). Ask groups to go back to their pictures and carefully draw the circles corresponding to the triangles in their Delaunay (tunnel) diagram. Ask them what they observe. Groups can compare to their diagram to others' diagrams and try to prove their claim.

The conclusion in mind is the following: any circle defined by a Delaunay triangle will not have any other points (houses) inside of it. This is called the 'empty ball' property. There is an option in the second link above to "Show Empty Circles" which also exhibits this property.

As groups finish, ask challenge questions such as:

- Where are the centers of the 'empty circles'?
- What positioning of houses makes for a very big circle or a very small circle?
- Draw four houses at the corners of a square. What do the circles look like? What happens if you move one corner 'out' a little bit? What happens if you it 'in' a little bit?

Closing remarks Discuss applications mentioned in abstract - show results of a Google Image search for "Voronoi Diagram".

Note to instructors: The case of houses placed at the corners of an obtuse triangle is important and worth doing in detail. Of the four examples, example 1 was very easy and 4 was reasonably easy, while 2 and 3 were the most challenging for the participants. You can easily make up your own examples using the above links or any Voronoi generator.