HERCULES AND THE HYDRA

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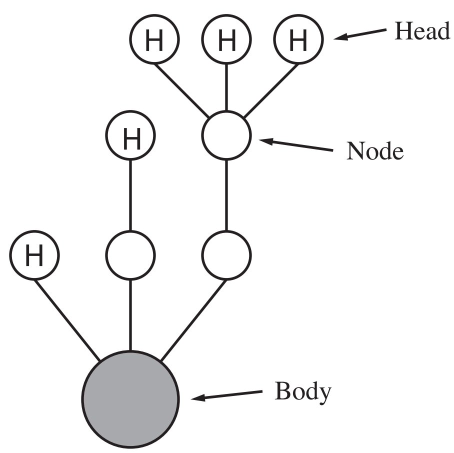
Supplies: Paper and pen(cil)

1. The Problem

After a late night reading about classical mythology (or watching "Clash of the Titans" yet again), you drift off to sleep and dream that you are face-to-face with a many-headed monster that is clearly not happy to see you, either. You look down and realize that you are wearing a lion skin and carrying a club. "Ahah," you think, "I must be Hercules and that . . . thing must be the hydra!" While you are thinking this, the hydra nearly takes your arm off. Then things get worse: you notice that *this* hydra has a tattoo of Pythagoras' theorem on its arm and that its heads do not all grow directly out of its body: some of its necks branch like a tree before ending in heads. (Some language: let's call the branching points on the neck "nodes")



Oh, no! A *mathematical* hydra! You recall from your misspent youth in Greek Heroes' Academy that the heads of a mathematical hydra obey the following rules:

(1) If you cut off a head attached directly to the body, the head dies and nothing further happens.

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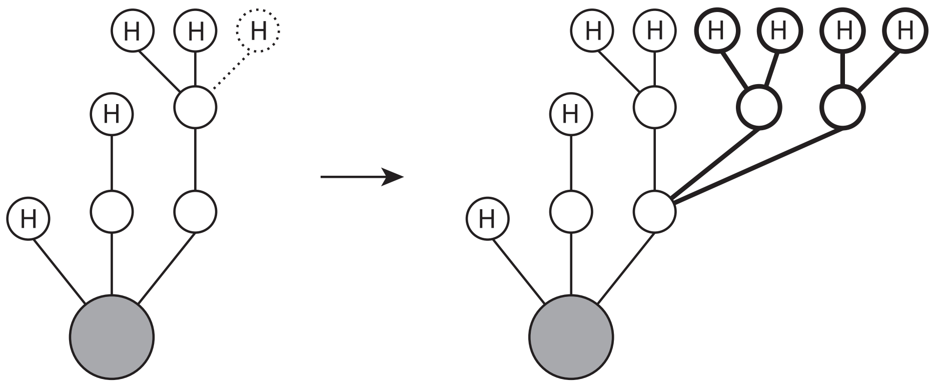
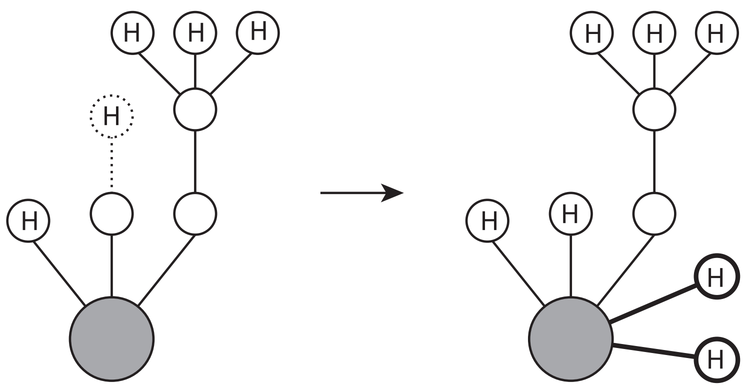
This problem is based on a theorem of Kirby and Paris. This session was prepared with the cooperation of the Philadelphia Area Math Teachers' Circle Leadership Team: Catherine Anderson, Kathy Boyle, Aimee Johnson, Amy Myers, and Josh Taton.

(2) If you cut off a head, new heads are grown as follows: look at the node from which the head grew. Move one node closer to the body of the hydra. Finally, from the node closer to the body, grow back two copies of the part of the (decapitated) hydra that starts at the node from which the head grew.
(3) If a node has no "outgoing" necks, then it becomes a head.
(4) The hydra dies when it has no more heads.





Question 1. Can you kill this hydra? What hydras can you kill?

2. Extensions to the Problem

Question 2. Instead of growing *two* identical trees where the old tree was cut off, what happens if the hydra grows *three* identical trees? Does your strategy still work? What if the first head that gets cut off grows back one additional tree, the second grows back two trees, the third grows back three trees, etc.?

Question 3. Make up your own question about hydras!

These questions could be further generalizations of the first extension (n trees grow back, or a random number of trees grow back), or could be questions about optimal strategies, or even for counting the number of steps for specific types of hydras . . .

3. Connections to Higher Mathematics

Secretly, it turns out that *any* strategy — even randomly cutting off heads — will eventually work! This is Kirby and Paris' result, and they use some deep ideas from mathematical logic (especially "ordinal numbers") in their proof.

4. Common Core Standards

The focus of this problem is on the Standards of Mathematical Practice:

MP1: Make sense of problems and persevere in solving them.
MP2: Reason abstractly and quantitatively.
MP3: Construct viable arguments (especially reasoning inductively).
MP8: Look for and express regularity in repeated reasoning.

5. Sources

- Kirby, L. and Paris, J."Accessible Independence Results for Peano Arithmetic". Bulletin of the London Mathematical Society 14 (1982), pp. 285+.
- Forisek, Michael. "What is an example of a counterintuitive result in mathematics?" `http://www.slate.com/blogs/quora/2014/06/04/hydra_game_an_example_of_a_counterintuitive_mathematical_result.html`, accessed July 2, 2014.