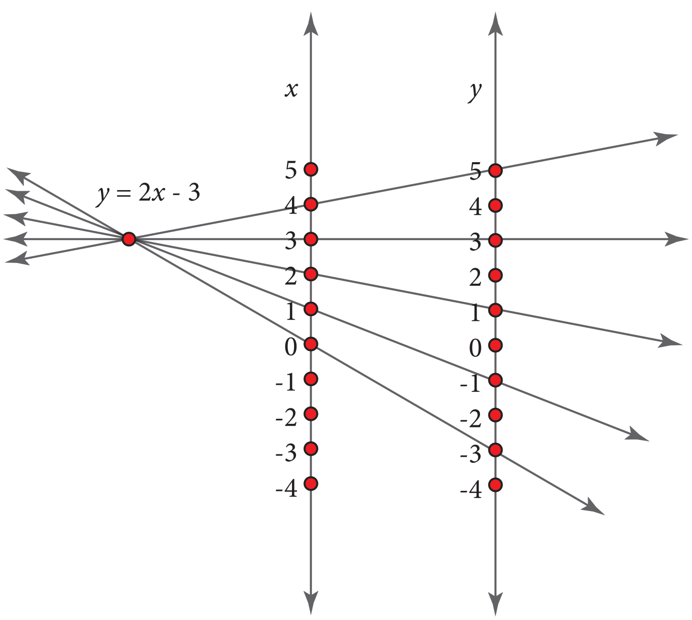
In Session: Function Diagrams

A Session by Henri Picciotto

Some of the best sessions come from what-if questions. What if Euclid's fifth postulate weren't true? What if only even numbers had been invented?[1] What if we graphed with parallel axes instead of perpendicular ones? What would that even mean?

Let's begin by graphing $y = 2x - 3$. We do this in the ordinary way: find some pairs of values that work, like (0, −3) and (1, −1) and (2, 1) and (3, 3) and (4, 5). But now, instead of plotting a point for each of these pairs, we locate the x value on the x axis and the y value on the y axis, and then connect them with a line. If we tried to plot all the pairs, we'd end up with infinitely many lines and it would look like a solid blob of ink, so we plot just a few pairs to illustrate the pattern.



And look at what we discover! The graph of the equation $y = 2x - 3$ is a point: the point where all the lines (0, −3), and (1, −1), and so on, meet. You might enjoy finding a proof that all these lines really do meet at a point, called the focus. More importantly, before you read farther, graph a few other equations on parallel axes. It might be convenient to place your axes 6 units apart. Henri Piccotto's first handout[2] might also make some good practice here, especially with the teachers' notes that he gives[3].

There is a big principle at work here: duality. In many geometries, there are interesting transformations that exchange points and lines, and replace points being on the same line with lines passing through the same point as we see in this example. Using that idea, can you figure out how to use function diagrams to solve a system of equations?

Perhaps more importantly, we now have a new representation for our old friend $y = mx + b$. A new representation will make different aspects more salient; we'll tend to notice new patterns. For me, the big difference with these function diagrams is that I now notice the horizontal lines in them: that is, where the input x value is equal to the output y value. I would be prone to write our sample equation as $y - 3 = 2(x - 3)$ instead, to emphasize that fact. To practice writing equations for different function diagrams, you can download another handout[4], and also enjoy the teachers' notes that go with it[5].

The similar triangles that you can see in this representation also give a meaning to the m in our equation: It's the magnification! Each unit on the x-axis gets magnified by a factor of m as we go to the y-axis. It may take a while to get a gut-level appreciation that the magnification stays constant even while the "slope" of these lines is varying.

A new representation can also make certain aspects of our equation $y = mx + b$ easier to see and understand. For example, consider the amount of medication

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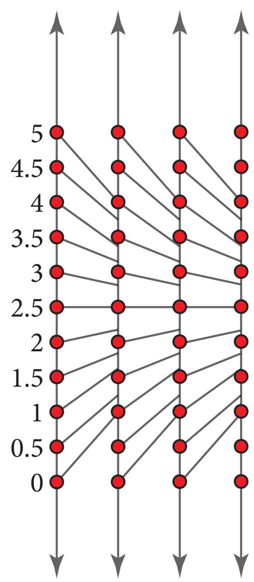
flowing in the bloodstream of someone who takes one antibiotic pill per day. Perhaps the body removes 40 percent of the medicine in a day, and then 1 more unit is added, so with y being the next day's amount and x being the current amount, we have $y = 0.6x + 1$. If we want to know how much of the medication will be in the bloodstream a few days later, we can use an iterated function diagram like this one:



The function is the same every time, but since we're only plotting a few of the (input, output) pairs, you'll have to imagine the way things flow in the spaces between the lines as we apply the function repeatedly.

Now we can see that, no matter where you start, after a few days you'll already be pretty close to 2.5 units of medication. And now you know why some antibiotics have you take two pills the first day, too!

This representation also lends itself to the composition of functions. A few such diagrams and some thought about magnifications will lead to an easy, visual proof of the chain rule for derivatives.

You may also make some interesting discoveries of your own by graphing quadratics. There's an amazing thing to notice in the graph of $y = 1/x$, too. Enjoy!

References

1. "**Primes in Evenland**," New York Times Wordplay blog.

2. "**Nine Function Diagrams**," Math Education Page.

3. "**Function Diagrams**," by Henri Picciotto, Math Education Page.

   (Author's note: See also the additional links at the bottom of the web page. The brief history and bibliography there cite many sources for these ideas. If I recall correctly, I first learned about function diagrams at a conference session led by a couple of teachers from the Illinois Math and Science Academy.)

4. "**Sixteen Function Diagrams**," from *Algebra: Themes, Tools, Concepts* by Anita Wah and Henri Picciotto, Math Education Page.

5. "**Focus on Function Diagrams: Teacher Notes**," from *Algebra: Themes, Tools, Concepts* by Anita Wah and Henri Picciotto, Math Education Page.

For links to these resources and more, visit us online at ***http://mathteacherscircle.org/resources/sessionmaterials.html***

Figures: Jessa Barniol