Analyzing Unique-Matching Games Using Elementary Mathematics

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Introduction

We present here an analysis of the game of ***Spot It!*** that can be used as background information for the purpose of leading a math circle around the game. In this paper we present an algorithmic/numerical approach that is complementary of the geometric approach given in [1]. The presentation here follows the progression of solving the puzzle for simple cases first and culminating in the solution of the original game.

Game Instructions

Blue Orange Games' award-winning ***Spot It!*** is a card game for 2 to 8 players, ages 7 to adult. Additional versions, such as ***Spot It! Numbers and Shapes***, are for fewer and younger players. Players must match a symbol on their card with the top card of the draw pile. The first person to spot a match gets that card, which then becomes the card to match with the next card in the draw pile. The person with the most cards when the draw pile is gone is the winner.

Game Details

Each card contains 8 symbols, and any two cards in the deck share *exactly one symbol*, though these need not be present at the same size. An instruction sheet included with the game exhibits 21 example figures from the pool of symbols appearing on the cards. It doesn't tell the total number of symbols used, though it does give the number of cards in the deck — but let's leave that unknown for the moment, especially since the company didn't produce as many cards as they could have.

Main Puzzle to Solve

Determine for 8 distinct symbols appearing on each card just how large a pool of distinct symbols can be used and how many cards can then be made so that any pair of cards shares exactly one symbol and so that all matching possibilities are realized by the set of cards.

Method of Solution

We will explore some simpler setups before solving the main puzzle and generalizing. Acting this out like a manufacturer, we first specify a value for the number of symbols s to place on each card (say, $s = 3, 4, 5, \ldots$) and then see what choices there are for the size p of the symbol pool and the number of cards c. This seems to be the right approach for a more theoretical analysis as well: given a value s, determine unique optimal values (if there are such) for p and c.

Exploration

Begin by playing a ***Spot It!*** game with several players. Briefly discuss any mathematical questions that arise in connection with the game: *how many symbols are on each card? how many different symbols are used in all? how many cards are in the game?*

Next, to explore in more detail the quantitative thinking underlying the construction of the game, consider what the game might look like if a smaller number of symbols had been placed on the cards. Start with $s = 2$ or $s = 3$, and then move on to larger values of s, using your explorations to conjecture how p and c might be related to s in general.

Note, there are some "boring" solutions that we will implicitly ignore here. For example, one could print a deck in which every card has a star on it, and all the remaining symbols are unique and appear exactly once. This would be a very boring game, so we are looking for a solution that is "fun" to play. This assumption can be made mathematically precise as is done here [1].

Case 2: s = 2

It is easy to determine by trial and error that if $s = 2$, the optimal values for p and c are both 3. Using numerals for our symbols, the three cards in the deck would be $(1, 2)$, $(1, 3)$, and $(2, 3)$. This may still be a boring game, but it's a base case for a more interesting game.

Case 3: s = 3

This case is also fairly easy but not immediately obvious. Here the optimal values for p and c are both 7. One can construct a deck of 7 cards, each containing 3 symbols drawn from a pool of 7 symbols so that every pair of cards shares exactly one symbol and so that all possible matches are realized.

We'll order these cards by first listing those that match a common symbol; without loss of generality, let this symbol be 1 and call this list of cards the 1-list. The optimal 1-list is as follows:

$$\begin{array}{ccccccc} 1 & 2 & 3 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & 4 & 5 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & 6 & 7 \end{array}$$

These are the only cards that can match 1's. For no new card can contain a 1 along with a number $n \leq 7$ or it will match two symbols with one of the given cards. Nor can a new card contain a 1 along with two new symbols $8, 9$; else any additional card with smaller numbers not in the 1-list, such as (2,4,6), will fail to match such a card. So the maximum number of cards in the 1-list is 3 and the maximum value for the pool of symbols is $p = 7$.

This also means that 3 is the maximum possible number of cards that can match in *any given symbol* n: there's nothing special about matching the numeral 1. Since all the numerals $n \neq 1$ appear once in the original 1-list, n can only appear twice more. This occurs already in the 2-list and 3-list cards, so the maximum number of total cards is $c = 3 + 2 \cdot 2 = 7$.

The full list of cards is given below. We begin by repeating the initial card and the 1-list and then append the additional 2-list and 3-list cards that can be taken.

1	2	3	·	·	·	·
1	·	·	4	5	·	·
1	·	·	·	·	6	7
·	2	·	4	·	6	·
·	2	·	·	5	·	7
·	·	3	4	·	·	7
·	·	3	·	5	6	·

Here our first 2-list card is (2,4,6). This choice dictates what follows. The other 2-list card must be (2,5,7). Next, we can't choose (3,4,6), so we must have (3,4,7) and (3,5,6) as the last cards. This yields a maximal list of 7 distinct cards, each pair of which satisfies the game's requirements.

Case 4: s = 4

Having explored the cases $s = 2$ and $s = 3$, this next case isn't too difficult; much of what we said above extends to $s = 4$. We could more or less find the solution by trial and error, and that might be true in this case as well, but it does get more difficult so it will be well to search for a few general properties the deck must posses first.

- There are at most 4 distinct cards in the optimal 1-list.
  Similarly, then, for the lists that match 2, 3, 4, or any later numeral.
  - *Proof*: suppose there are m cards in the 1-list.
    Then any card *not in the 1-list* must contain at least m distinct symbols to match the 1-list cards properly: it can't contain less, or two cards in the 1-list will have a double

match.
  This means $m \leq 4$; there are at most 4 cards that contain a 1 (similarly for any symbol).
- The maximal number of cards in the deck equals the size of the symbol pool: $c = p$.
  - *Proof*: the total number of symbols that can appear on all the cards is $4c$.
    The number of distinct symbols among these can be found by factoring out the duplication. Assuming each symbol appears four times we have: $p = 4c/4 = c$. If symbols appear fewer than four times this can only further limit the number of cards, so the largest possible deck must have the same number of symbols as cards.
- The maximal number of cards is also given by the formula $c = \binom{p}{2} / \binom{4}{2}$.
  - *Proof*: the total number of symbol pairs possible is $\binom{p}{2}$.
    No two cards share the same pair of symbols or there would be a double match.
    Thus, $c \cdot \binom{4}{2}$ distinct symbol pairs can be formed from symbols on a common card.
    If this is as large as possible, then $c \cdot \binom{4}{2} = \binom{p}{2}$; i.e., $c = \binom{p}{2} / \binom{4}{2}$.
- The maximal pool of fully matchable symbols is $p = 4 + 3 \cdot 3 = 13$.
  - *Proof* #1: we can flesh out 1-list cards after the first one $(1, 2, 3, 4)$ by successively appending 3 distinct triples, yielding cards $(1, 5, 6, 7)$, $(1, 8, 9, 10)$, and $(1, 11, 12, 13)$.
    Any additional card, such as $(1, 14, 15, 16)$ will fail to match some viable card containing smaller numbers, such as $(2, 5, 8, 11)$.
    So there are at most $4 + 3 \cdot 3 = 13$ symbols in our symbol pool.
  - *Proof* #2: Alternatively, we know that $p = c = \binom{p}{2} / \binom{4}{2}$.
    Thus, $p = \frac{p(p-1)}{12}$; i.e., $12p = p(p-1)$.
    Cancelling p, we conclude $p = 13$.

Of course, we need to show that this potential number of possibilities can be realized. Our approach is constructive and generalizes what we did above: we list all the working possibilities via a uniform procedure that passes through blocks of the new numerals in a shifting diagonal fashion. Here is our list of 13 cards organized into n-lists for $n = 1, 2, 3, 4$.

1	2	3	4	·	·	·	·	·	·	·	·	·
1	·	·	·	5	6	7	·	·	·	·	·	·
1	·	·	·	·	·	·	8	9	10	·	·	·
1	·	·	·	·	·	·	·	·	·	11	12	13
·	2	·	·	5	·	·	8	·	·	11	·	·
·	2	·	·	·	6	·	·	9	·	·	12	·
·	2	·	·	·	·	7	·	·	10	·	·	13
·	·	3	·	5	·	·	·	9	·	·	·	13
·	·	3	·	·	6	·	·	·	10	11	·	·
·	·	3	·	·	·	7	8	·	·	·	12	·
·	·	·	4	5	·	·	·	·	10	·	12	·
·	·	·	4	·	6	·	8	·	·	·	·	13
·	·	·	4	·	·	7	·	9	·	11	·	·

We have listed these cards in a chart using parallel diagonals within each n-list, but we can also specify these cards more analytically. This gives us a second way to think about our cards, which we will spell out in full detail when we investigate the original ***Spot It!*** game (Case 8).

Observations on Specifying and Matching Game Cards for s = 4

Individual Card Values

- Each card contains 4 distinct symbols

  -For the 1-list, new symbols are added in blocks of three to produce each additional card.

  -For later n-lists, the three additional values on each card after the initial n are chosen in order, one from each of the original three new-symbol blocks, yielding four distinct values.

Matching Cards

- Pairs of n-list cards match in the numeral n and no other numerals.

  -Each n-list card for $n > 1$ begins with n, which they all share.

  -No later numerals are shared by different cards in a common list because the numbers in each block proceed along a diagonal to guarantee distinct values for different cards in those spots.

  -To be precise, different cards are finished off differently by starting with *distinct* initial values and then continuing in the *same way*, using a delayed mod-3 addition (i.e., use 3, not 0) to run through all possible positions for each card.

  Ex: all of the cards in the 4-list begin with 4; the first entry after 4 on card m is the m^{th} entry of the first block $(5, 6, 7)$; the second entry is the $(m+2)^{\text{nd}}$ entry of the second block $(8, 9, 10)$; and the third entry is the $(m+4)^{\text{th}}$ entry of the third block $(11, 12, 13)$, where the value of the entry-positions in the blocks are calculated using delayed mod-3 addition.

  Thus, the 3^{rd} card in the 4-list is $(4, 7, 9, 11)$, with the entries after 4 appearing in positions 3, $2 \equiv_3 3+2$, and $1 \equiv_3 3 + 2 \cdot 2$ within their respective blocks (here $m = 3$, and the repeated addend is 2).

- Proper matching between cards in different n-lists can be made by comparing each card to all the cards above it in other lists. This is easily done using the chart provided, though it's a bit time-consuming and tedious when done by hand. Clearly, a better method of comparison is required for larger values of s, as well as for generalizing! We'll exhibit this for Case 8.

Case 5: s = 5

For this case, we can still argue that $p = s + (s-1)^2 = c$ gives the maximum number of symbols and cards; i.e., $p = 5 + 4^2 = 21 = c$. However, this doesn't mean these numbers can be realized in a unique-matching game. In particular, the diagonalization method we used above will not produce 21 cards from 21 symbols that share only one symbol per pair (try it). The problem here arises from the fact that $s-1 = 4$ is not prime. In this case, however, it is possible to reorder the numerals on the cards so that 21 properly matching cards can still be produced (homework project).

Case 6: s = 6

This case applies to the ***Spot It! Numbers and Shapes*** junior version of the game.
Since $s = 6$, $p = 6 + 5^2 = 31$ calculates the maximal number of symbols that can be used for such a game. Furthermore, since $s - 1 = 5$, which is prime, the same diagonalization procedure as used above for $s = 3$ and $s = 4$ will produce the maximum number of $c = 6 + 5^2 = 31$ distinct cards that match each other in the way required (try it). This analysis agrees with the number of cards produced by Blue Orange Games for their game. One solution is given in figure 1, though matching the numbers to the symbols on the cards may be interesting.

Checking that these cards match appropriately can be done as indicated above for Case 4; however, we're now reaching the point where an even more systematic method may save us some time: do you really want to check ${}_{31}C_2 = 465$ pairs? This number can be reduced if we grant, as above, that n-list cards share only n by construction and that we only need to check whether each card properly matches earlier cards in other lists after the 1-list, but this still leaves quite a few pairs to check (a mere 250 now). We'll explore this matter more analytically when we work through the next case that succumbs to our diagonalization procedure, Case 8.

Case 7: s = 7

This case cannot be solved by using a diagonalization process, just as with Case 5, because $7-1 = 6$

is not prime. However, this time there is no other way of juggling the numbers on the various cards to create a ***Spot It!*** game with 7 symbols. This follows from results on finite projective planes. Treating cards as "points" and sets of cards matching a particular symbol as "lines", the game forms a projective plane; see [1] for a thorough analysis of the game from this perspective. The diagonalization approach here is designed to be accessible to someone with a more elementary knowledge of mathematics. In short it has been proven that this is impossible; see [2] for details. Interestingly it was also shown through a massive super-computer search that case $\mathbf{s} = \mathbf{11}$ is also impossible. It is an open problem, currently even beyond the reach of supercomputers, whether a game of ***Spot It!*** with $\mathbf{s} = \mathbf{13}$ is possible.

Case 8: $\mathbf{s} = \mathbf{8}$

This case finally addresses the original ***Spot It!*** game, in which each card contains 8 symbols.

It can be argued, as above, that the maximal values for the number of symbols and cards is given by $p = 8 + 7^2 = 57 = c$. Each n-list after the original card $(1, 2, 3, 4, 5, 6, 7, 8)$ contains at most 7 cards, and each symbol can appear at most 8 times among these cards.

The more difficult thing to demonstrate is that one can create this maximal number of cards[1] using the maximal number of symbols so that each pair of cards matches in exactly one symbol. The diagonalization procedure described above will produce 57 cards that suitably use 57 symbols (an easy, if time-consuming, construction task). However, using the chart so created to prove that these cards match properly is tedious; now at least 1029 pairs of 8-symbol cards need to be checked.

We therefore take a different tack by exhibiting formulas for the various cards and then show how pairs of cards match up. The following assertions can be checked against the chart listing the cards, which is appended in figure 2.

Formulas for Card Entries when $\mathbf{s} = \mathbf{8}$

Individual Card Values

- Each card contains 8 distinct symbols
  - -For the 1-list, blocks of seven new symbols are appended to produce each additional card.
  - -For later n-lists, the seven additional entries on each card after the initial n are chosen in order, one from each of the seven new-symbol blocks, obviously yielding seven distinct values.
  - -To be more precise, card m in each n-list (for $n \geq 2$) has its successive entries after n in the following block-positions, calculated as delayed mod-7 values: $m + 0k, m + k, \ldots, m + 6k$ for $k = n - 2$. In fact, these block-positions run through all possible values and are distinct because 7 is prime; if $m + ik \equiv_7 m + jk$, then $i = j$.

  Ex: $(8, 13, 19, 25, 31, 37, 50, 56)$ = card 5 in the 8-list has its distinct entries after 8 in block-positions $5, 4, 3, 2, 1, 7, 6$ for the various blocks of numerals (here $n = 8$, $m = 5$, and $k = 8 - 2 = 6$ = the repeated addend used in the delayed mod-7 addition).

Matching Cards

- Each pair of n-list cards shares only the numeral n.
  - -The 1-list cards obviously agree in only 1 since they are finished using different blocks of new numerals.
  - -Each later n-list card begins with n, which they all share. No later numerals agree on different cards because they start with different initial position values and continue with entries drawn from different positions in successive blocks, whose values are determined using repeated delayed mod-7 addition of the same constant.

    More precisely, cards m_1 and m_2 in the same n-list have their entries after n in the $m_1 + ik$

[1] Curiously, there are only 55 cards for 57 symbols in the ***Spot It!*** game. This was for marketing purposes and due to limitations from standard printing practices, see [1]. Homework: which two cards are missing from the official deck?

and $m_2 + ik$ block-positions, where $k = n-2$, $i = 0, 1, \ldots, 6$, and sums are evaluated delayed mod 7. If $m_1 \neq m_2$, all $m_1 + ik \not\equiv_7 m_2 + ik$. Thus, distinct cards have entries drawn from distinct block-positions throughout, making them all distinct.

Ex: card m in the 5-list is filled out by starting with the entry in position m for the first block and then repeatedly adding $k = 5 - 2 = 3$, delayed mod 7, to this value to determine later block-positions for entries. Thus card 6 = $(5, 14, 17, 27, 30, 40, 50, 53)$ has its entries after 5 in positions $6, 2, 5, 1, 4, 7, 3$; while card 2 = $(5, 10, 20, 23, 33, 43, 45, 56)$ has its entries after 5 in positions $2, 5, 1, 4, 7, 3, 6$—the same cyclic list of all positions, but starting differently.

- Two cards in different lists match uniquely in the entry located in a shared block-position.

  To be more precise, card m in the n-list matches cards m, $m + d$, $m + 2d$, $\ldots$, $m + 6d$ in the earlier $(n - d)$-list with its successive entries after n, calculated delayed mod 7.

  To spell this out in more detail, where all calculations are done via delayed mod-7 arithmetic:

  - The first entry $[i = 0]$ after numeral n of card m in the n-list is in block-position m, which matches card m in all lists.
  - The next entry $[i = 1]$ is in position $m + (n - 2)$ for the n-list card, which matches card $m + d$ of the $(n - d)$-list in block-position $(m + d) + (n - d) - 2 = m + (n - 2)$.
  - Generalizing, entry i after n on card m of the n-list is in block-position $m+i(n-2)$, which matches the block-position of entry i for card $m+id$, which is $(m+id)+i((n-d)-2) = m + i(n - 2)$.
  - To see that these matches are proper, first note that since 7 is prime, *all* cards in the $(n - d)$-list match up (differently) with card m from the n-list. Now suppose that card m of the n-list matched some card in the $(n - d)$-list in more than the place specified. Then this value would be shared by two $(n - d)$-list cards, which can't happen, since they match properly. Therefore, the matching specified between card m of the n-list and all earlier $(n - d)$-list cards is proper. But as these are the only cards that card m of the n-list needs to be compared with, our listing shows that all pairs of cards have been appropriately matched.

  Ex: $(8, 13, 19, 25, 31, 37, 50, 56)$ = card 5 in the 8-list matches cards $5, 3, 1, 6, 4, 2, 7$ in the 3-list with its successive entries; here $m = 5$, $n = 8$, and $d = 8 - 3 = 5$.

  Matching card 4, say, in the fifth entry after 8, it cannot match any other 3-list card in that entry, or two 3-list cards will share two symbols, both 3 and 37, which they don't.

Conclusion

As was mentioned before there are many choices for N in which no deck can be built, however the algorithm presented for $s = 8$ can be generalized to a case when $s - 1$ is prime. Though we will not include such a proof here as it is beyond the scope of this paper and likely beyond what is important for teachers. What is particularly interesting about the game of ***Spot It!*** is how it is simple to explain and play (it's made for kids) and yet is full of very wonderful and deep mathematics. It's remarkable that the game naturally leads to open questions in mathematics. This presents a wonderful opportunity to help non-mathematicians understand what "research" is and how simple problems can yet be unsolved.

Appendices

1	2	3	4	5	6																									
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Figure 1: Spot It Solution For 6 Symbols

1	2	3	4	5	6	7	8	Block i = 0							Block i = 1							Block i = 2							Block i = 3							Block i = 4							Block i = 5							Block i = 6						
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Figure 2: Spot It Solution For 8 Symbols

References

[1] Deepu Sengupta (2016), "A Mathematical Analysis of Spot It!," `http://www.mathteacherscircle.org/assets/legacy/resources/materials/DSenguptaSpotIt.pdf`

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