Art Meets Math: Escher's Tilings

by Altha Rodin

The Dutch artist M. C. Escher is well known for his amazing prints of interlocking lizards, fish transforming into birds, and angels and devils intertwined, just to name a few. His intricate tilings offer a beautiful and engaging way to explore ideas related to geometric transformations and symmetries.

Escher himself is an example of someone who used mathematics extensively in his work, although he never considered himself a mathematician. By his own admission, he was unenthusiastic about his mathematics classes and found the abstract aspects of mathematics daunting. And yet, in his career as an artist, he operated as a mathematician, posing questions and investigating ideas until he reached satisfactory answers to the problem of putting down on paper what he saw in his imagination. He is, in many ways, a model of how we want our students to approach problems they encounter in the classroom and beyond.

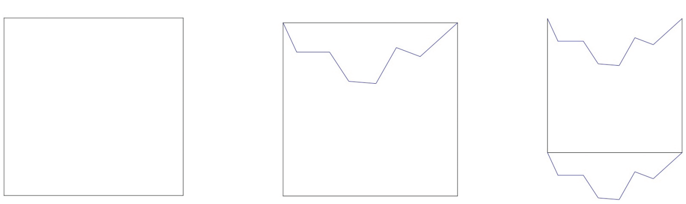
First, a definition: A tiling, or tessellation, of the plane is a covering without gaps or overlaps, by congruent copies of one or more shapes. Escher's tessellations are often composed of intricate designs, but underlying these shapes are very simple tilings of the plane by regular polygons. A good way to begin a session on Escher-like tilings is to start with an examination of these underlying tilings. The simplest tilings to categorize are regular tilings in which all the shapes are congruent to a single regular polygon, and all polygons meet along complete edges. The question of finding all regular polygons that can be used in a regular tiling gets participants thinking about properties of regular polygons, and in particular about the relationship between the number of sides of a regular polygon and the measure of the interior angles. It does not take long to realize that there are only three regular polygons that will produce a regular tiling. At this point, it may be a good idea to ask for volunteers to explain how they know for sure that the three regular tilings they found are the only ones. It is not hard to justify this result, but sometimes we all need to be reminded that even things that seem obvious can and should be justified.

A good follow-up question, once the problem of finding all regular tilings has been settled, is to consider how the restrictions on regular tilings can be relaxed and whether the resulting tilings can be classified in some way. This is a very broad question and a whole session could be dedicated to it. The session leader may want to leave all options on the table and see where the questions that the participants come up with will lead. Alternatively, the participants can be directed toward a particular type of tiling. In the session we did on tilings with the Math Teachers' Circle of Austin, we chose the latter approach, defining semi-regular tilings and leaving the problem of finding all semi-regular tilings as a contest problem for participants to think about on their own. The necessary definitions and statement of the problem are given in the handout from that session, which is available on the **MTC website**.

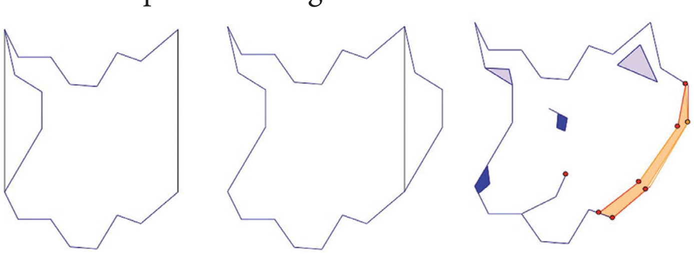
The key idea in producing Ecsher-like tilings is to modify each tile in such a way that the modified tile still tessellates. This is done by altering the tiles in a way that reflects the symmetries of the desired tiling. Many teachers and students alike have an intuitive understanding of what a symmetry is and can recognize symmetry when they see it, but to really understand symmetry in the way we want to use it, it is necessary to understand rigid motions of the plane, i.e., motions that preserve distances. There are four types of rigid motion of the plane: reflection across a line, rotation about a point, translation by a vector, and glide reflection. Most people are familiar with the first three. A glide reflection is simply a composition of a reflection across a line with a translation parallel to the line of reflection. A symmetry can be described as an undetectable rigid motion. For example, a square has reflectional symmetry across a diagonal because, if someone were to reflect across that diagonal while you were not looking, you could not tell that the reflection had occurred. After introducing the idea of rigid motions, it is a nice exercise to examine some of Escher's drawings to identify the symmetries. It is interesting to consider how the symmetries are affected by coloring. As a general rule, an uncolored tiling will have more symmetries than a colored one.

Finally, it is time to get artistic. The easiest Escher-like tiling to create is one based on a square tiling of the plane, using the translational symmetry of the square tiling.

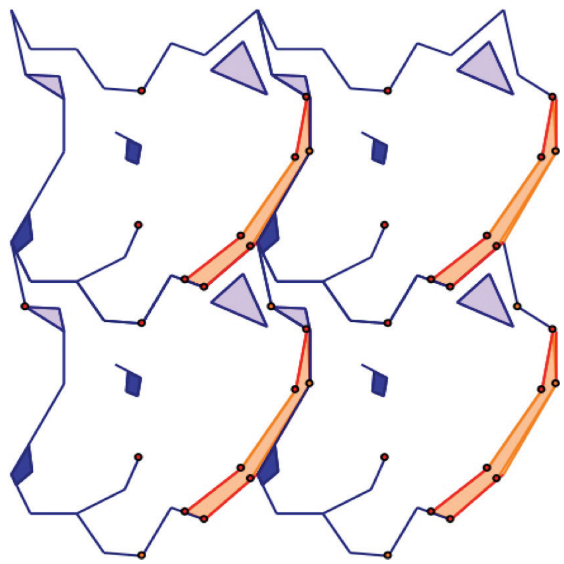
Start with a square cut out from an index card, approximately 2 inches in length, and cut a piece off of the top edge. Tape the cutout piece to the bottom edge of the square. You have translated the modification of the top edge to the bottom.



Do the same on the sides: cut a piece off of the left side and tape it to the right side and decorate.



Now you are ready to tile. Trace around the template you have made and add details.



The tiling produced in this manner has the same translational symmetries as the regular tiling by squares, but it no longer has the other symmetries, like rotational or reflectional symmetries, of regular tiling.

Next, participants can be asked to experiment with ways in which the sides of the square can be altered to maintain other symmetries of the square tiling. Alternatively, other regular tilings or tilings by parallelograms, which are often at the base of Escher's tilings, can be used as the starting point for an Escher-like tiling. The possibilities are endless, the resulting tessellations are aesthetically pleasing, and if all goes as planned, the participants walk away with a new appreciation for geometric symmetry.

For those interested in learning more about the mathematical side of Escher, check out the article "Escher as a Mathematician," by Doris Schattschneider, available on the **AMS website**.

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