Math Teachers' Circle
Coins in Twoland

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In Twoland, the only money is coins with value 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, and so on.

There is also a law that requires all purchases be made with exact change. Only the banks are allowed to make change.

Oneville

In Oneville, there is an additional law: purchases must be made with at most one copy of each coin. So for instance, paying 5 must be 4 + 1 (or 1 + 4, but we'll consider that the same). It would be illegal in Oneville to hand over 2 + 2 + 1 in payment.

1. How many ways are there to pay 1023?

2. How many ways are there to pay 1024?

3. How many ways are there to pay 1000?

Where do we live?

4. How many ways are there to write 349 in our system? Of course we can't write $2_{14}9$ with a 14 in the tens place - why not? What are the laws?[1]

Twoville

Twoville, the capital of Twoland, turns out to be a much more interesting place. Their law requires that payments be made with at most two copies of each coin. So now you can pay 5 with 4+1 or 2+2+1, so there are two ways to do it.

5. How many ways are there to pay 1023?

6. How many ways are there to pay 1024?

7. How many ways are there to pay 1000?

8. How many different methods can we find for solving that last problem?

[1] We live in Tenland, in the town of Nineville.

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I think these problems can be used fairly directly with students. 1000 might be just a little bit large for them, but if you go too much smaller they will solve it with brute force instead of figuring out a general method that will let them work it out for any number.

Depending on their level, you might give them some worksheets that help them organize their efforts: a table of values with space to write a list of ways, for instance.

For some bigger hints, you might give them a table that's organized into rows for each power of two, so they write the number of ways to make 1 on the first line, 2 and 3 on the second line, 4, 5, 6, and 7 on the third line, and so on.

You might also suggest that they categorize the ways into groups depending on the number of 1s they use. Of course they'll quickly see that all the odd amounts require exactly one 1, and the even amounts require 0 or 2. Perhaps then they can discover the ways those amounts relate to previous cases.

I find that kids are often interested in understanding why our system of numbers actually works, with just one way to express each positive integer. Some kids may do better skipping over the Tenland problem, though, or exploring it differently. For example, ask about writing 399 as 40_{-1}, or in other words 400 + -1. Most everyone will object, but you can point out how much easier it is to add 273 + 399 if you write it as 273 + 40_{-1} instead.

Naming the value of the 1-unit coin something that is humorous to your local clientele is also very useful. Here I left it unnamed. “Cents” is a bad idea because then it should be hundredths of something. I like “Zlotys” because they will think you are making it up, but maybe it's better to use something really made up.