

Visualization and Symmetry (Unfinished)

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Abstract

We will examine visualization and symmetry in a very general way by means of a set of problems. Many topics in mathematics can be made much clearer when symmetric aspects are made clear or when nice alternative visualizations are possible. When this occurs, it helps both the student and the teacher.

There is a large amount of potential classroom material here, and almost any small part of it could be used for an entire class session.

1 Introduction to Visualization

All of us (including both our students and ourselves) think differently. Some of our brains work well manipulating symbols (algebraic computations, for example) and others of us are better at imagining and mentally manipulating shapes and diagrams (we'll call this geometric manipulation). Often problems can be viewed both algebraically and geometrically, and can be attacked using both methods.

We will use the term "visualization" here in a very general way. Basically, the idea is to try to find different ways to think about each problem since each different view gives us more understanding. The more different ways you have of looking at a problem, the better you will understand it.

Note: Some of the exercises below are marked with one or two asterisks: (*) or (**). These indicate problems that may be more difficult or much more difficult, respectively, than the others. Of course these are totally subjective determinations by the author; different people have different talents.

This article is still a work in progress, and not all the solutions are complete. Any problem that has a solution here will include a number, like "Solution: Item 5". This means that item number 5 in Section 5 is a solution (or at least a hint) for that problem.

2 Visualization of Algebraic Manipulation Rules

As a first example, we will try to come up with a set of ways that students can think about algebraic concepts in a way that makes them not just a set of somewhat arbitrary

rules, but as sensible ideas that are “obviously” true. There’s nothing special about the examples below, except that whenever we present a new topic in class, it’s good to try to think of ways that the new concept is “natural”, based on what the students already know.

1. **The distributive law.** Rather than just the sterile formula that states that given any three real numbers A , B and C that:

$$A(B + C) = AB + AC$$

why not approach it this way:

“The parentheses group things together. Suppose we’re thinking about a bunch of married couples, each of which consists of a man and a woman and we could write an ‘equation’ that looks something like this:

$$\text{couple} = (\text{man} + \text{woman}).$$

The parentheses indicate that the man and woman are grouped together. What would 8 such married couples look like? Well, it would be 8 copies of that group:

$$8 \text{ couples} = 8(\text{man} + \text{woman}).$$

But isn’t it obvious that this would amount to 8 men and 8 women? We’re making 8 copies of the group, so that would be 8 of everything in the group.”

Next you could look at something slightly more complex, like packs that consist of 5 baseball cards and one piece of chewing gum. What would 3 of those packs look like? Well:

$$3 \text{ packs} = 3(5 \text{ cards} + 1 \text{ gum}).$$

It should be clear what the resulting collection consists of; namely, $3 \cdot 5 = 15$ baseball cards and $3 \cdot 1 = 3$ pieces of gum.

2. **The commutative laws for addition and multiplication.** This can probably be done with pictures that look something like these:

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If we only talk about addition and multiplication, students may think, "Why even mention the commutative law? It's obvious." So look at some operations that are *not* commutative, like subtraction and division. What about exponentiation?

3. **Associative laws.** By looking at the combination of dots as in the examples above, the associative laws for addition and multiplication can be made clear. This is very easy for addition:

$$(2 + 3) + 4 = 2 + (3 + 4)$$

is equivalent to:

$$(\bullet\bullet + \bullet\bullet\bullet) + \bullet\bullet\bullet = \bullet\bullet + (\bullet\bullet\bullet + \bullet\bullet\bullet).$$

For multiplication, the product of three numbers can be viewed as 3D blocks of "dots". If we agree that $a \times b \times c$ refers to a block of width a , length b and height c , then the two groupings that the distributive law declares to be the same just amount to slicing the block in different orders. The same total number of dots remains the same.

As we did with the commutative law, it's a good idea to look at examples of operations that are *not* associative. Again, subtraction and division are good examples. What about exponentiation?

4. **Combining like terms.** When asked to take an expression and "simplify" it, the following expression:

$$2xy + 3xy^3 + z + 2xy^3 + 3z$$

is probably a far more frightening example than:

$$2 \text{ dogs} + 3 \text{ cats} + 1 \text{ bird} + 2 \text{ cats} + 3 \text{ birds}.$$

If we think of " xy " as "dog", " xy^3 " as "cat" and so on, the two "expressions" above are equivalent.

For students who may try to combine terms like $3x$ and $4xy$ since they share an x , you may be able to show them why this won't work because it should be obvious that there's no way to combine 3 "dogs" with 4 "doghouses": the terms have to be identical before you can sum the constants.

5. (*) Nice examples for the use of the commutative and distributive laws and combination of like terms can be sought in ordinary arithmetic. If the students remember how to add and multiply, these operations can be used as examples. If they're rusty, maybe the laws can help remember the operations:

$$7 \times 368 = 7 \times (3 \times 100 + 6 \times 10 + 8) \dots$$

Similarly, addition is combining like terms, then carrying is regrouping.

6. (*) Possible idea: could polynomial multiplication (in one variable) be made clear in terms of standard multiplication? For example, if we agree that $x = 10$, then the following are equivalent:

$$123 \times 456 = (x^2 + 2x + 3)(4x^2 + 5x + 6).$$

The only problem here is that polynomial multiplication does nothing about carrying. In a sense, it's a shame that carrying makes ordinary arithmetic multiplication more difficult than polynomial multiplication.

7. Can we come up with others?

3 More Visualization Exercises

Mathematics uses both the (symbol-manipulating) left brain and the (visual, geometric) right brain. Both are important, although the emphasis in elementary mathematics courses is generally on “left-brained” activity. Following are some exercises in visualization, some general, and some specifically aimed at problems related to elementary algebra.

What we will do in this section is look at “translations” of algebraic problems into possibly more easily-visualized geometric problems.

3.1 Adding Series of Numbers

1. Summing the basic series. We will work out this first example in detail. Following are more examples that can be approached in a similar manner.

As an example, suppose we need to find the sum:

$$1 + 2 + 3 + \dots + 100.$$

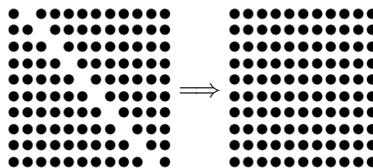
Let's look at a simpler example which, when solved visually, will make it obvious how to sum the series above. Let's add the following series visually:

$$1 + 2 + 3 + \dots + 10.$$

If we use “•” to represent a unit, then $1 = \bullet$, $2 = \bullet\bullet$, $3 = \bullet\bullet\bullet$, et cetera. Determining the sum from 1 to 10 is equivalent to counting the dots in the following pattern:



Just draw the same number of dots, but upside-down, and we obtain the following picture:



It's easy to count the dots in the pattern above: there are 10 rows of 11 dots, for a total of $10 \times 11 = 110$. This is twice as many as we want, however, so there are $110/2 = 55$ dots in the original triangular pattern.

2. Find the sum $1 + 2 + \dots + n$, where n is an arbitrary positive integer.
Solution: Item 1
3. Find the sum $1 + 3 + 5 + \dots + 1001$. In other words, sum the odd numbers from 1 to 1001.
Solution: Item 2
4. Find a general formula for the sum $1 + 3 + 5 + \dots + (2n + 1)$.
Solution: Item 3
5. Find the sum $7 + 10 + 13 + \dots + 307$.
6. Find the sum of a general arithmetic series:
$$a + (a + d) + (a + 2d) + \dots + (a + nd).$$
7. Find the sum of $1 + 2 + 4 + 8 + \dots + 128$. (Each term is double the previous.)
Solution: Item 9
8. Find the sum of $1 + 2 + \dots + 2^n$.
Solution: Item 9
9. Find the sum of $3 + 6 + 12 + 24 + \dots + 3 \cdot 2^n$.
Solution: Item 10
10. Find the sum of $1/2 + 1/4 + 1/8 + \dots + 1/256$.
Solution: Item 11
11. Find the infinite sum: $1/2 + 1/4 + 1/8 + 1/16 + \dots$.
Solution: Item 12
12. Find the sum of $a + ar + ar^2 + \dots + ar^n$.

13. (Telescoping series) Find the finite and the infinite sum below (each term has the form $1/(n(n+1))$):

$$1/6 + 1/12 + 1/20 + 1/30 + \cdots + 1/2550,$$

$$1/6 + 1/12 + 1/20 + 1/30 + \cdots .$$

Solution: Item 13

14. (*) Find the sum $1 + 4 + 9 + 16 + \cdots + n^2$. (***) Can you find the sum of the first n cubes? The first n fourth powers?

Hint and Solution: Item 14

15. Can you construct any nice formulas from the following pattern?

$$1 = 1^3$$

$$3 + 5 = 2^3$$

$$7 + 9 + 11 = 3^3$$

$$13 + 15 + 17 + 19 = 4^3$$

$$21 + 23 + 25 + 27 + 29 = 5^3$$

16. Can you use the diagram in Figure 1 to show another way to calculate $1^3 + 2^3 + 3^3 + \cdots + n^3$?

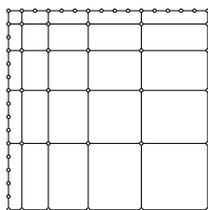


Figure 1: Sum of cubes

17. Try to draw a diagram with dots that demonstrates that:

$$(n+1)^3 = n^3 + 3n^2 + 3n + 1.$$

Hint: one nice solution is three-dimensional. What are the corresponding diagrams in two and one dimension?

Solution: Item 4

18. (*) Can we extend the idea above to help visualize something about four dimensions?

3.2 Equations and their Graphs

Next we'll look at the exact relationship between equations and graphs in two variables, x and y . The first couple of exercises seem unrelated, but will help get your mind thinking geometrically about the problems rather than algebraically.

1. Visualization exercises:

Try to solve these first in your head, without drawing pictures, if possible.

How many:

- corners (vertices) does a cube have?
- faces does a cube have?
- edges does a cube have?
- Same questions: how many vertices, faces, edges has a tetrahedron?
- ... has an octahedron?
- ... has an Egyptian pyramid?
- ... has a cube with a corner chopped off?
- ... have some other shapes?

Solution: Item 5

2. Describe:

- intersections of a plane with a sphere
- intersections of two spheres
- intersections of a cube with a plane

3. When we draw a graph of an equation like $y = 3x + 2$, *exactly* what does the graph mean? We often look at graphs of quadratic equations in particular for points where the curve (parabola, for quadratic equations) crosses the x -axis. What does this mean?

Solution: Item 6

4. Examine equations of lines, circles, and parabolas to see find some intuitive reasons why they have the form that they do.
5. What does the graph of this look like:

$$(x^2 + y^2 - 25)(3x - 2y + 3)(4x^2 - 3x + 2 - y) = 0$$

Solution: Item 8

6. Visualizing inequalities. For example, how are the graphs of $y = 3x + 4$, $y < 3x + 4$ and $y > 3x + 4$ related? How about $x^2 + y^2 < 25$?

7. Suppose we are looking for solutions to sets of simultaneous equations in two variables. The number of possible solutions can be imagined by manipulating the graphs in your mind. What sorts of situations can occur with equations having the following sets of graphs?
- two lines
 - three lines
 - line and circle
 - two circles
 - line and ellipse
 - two ellipses
 - parabola and circle
 - parabola and line
 - cubic curve and a line
 - cubic curve and a circle
 - two cubic curves
8. Making up equations of curves with given properties (like for an exam).
- Parabolas opening up or down. Left or right.
 - Parabolas symmetric about the y-axis.
 - A cubic polynomial that has roots 1, 2 and 3.
 - A cubic polynomial that has one root, two roots.
 - A line with slope $2/3$ that is tangent to the unit circle.
9. (*) Iterated functions.
10. Visualizing areas of geometric objects, from “first principles” – in other words, if all we know is that the area of a rectangle with sides of lengths a and b is ab , how can we derive the formulas for areas of objects like:
- a right triangle?
 - any triangle?
 - a trapezoid?
 - a circle?
11. (**) Same question as above, but in three dimensions: If all we know is that the volume of a rectangular solid is abc , where the lengths of the sides are a , b and c , how can we derive formulas for volumes of objects like:
- a pyramid?
 - a cone?
 - a sphere?
 - a prism?

4 Symmetry

Many of the problems below can be solved using standard, brute-force techniques, but every one of them can also be solved in a simpler way using a generalized notion of symmetry.

1. Given a standard 3×3 tic-tac-toe board, how many essentially different ways are there to make the first move?
2. (**) Given a $4 \times 4 \times 4$ three-dimensional tic-tac-toe board, how many essentially different ways are there to make the first move? (Hint: there is an “obvious” answer, but the real answer is surprising and amazing.)
3. (*) Given a pentagram with 10 holes as in Figure 2, fill in the holes with the following 10 numbers: 1, 2, 3, 4, 5, 6, 8, 9, 10 and 12 such that the sum of the numbers on each of the ten straight lines is the same. (Suggested by Harold Reiter.)

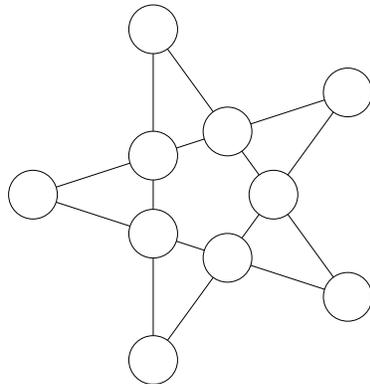


Figure 2: Fagnano's Problem

4. Suppose you decide to sell sudoku puzzles to your local newspaper, but you are too lazy to work out any actual puzzles, so your plan is to steal an existing puzzle and modify it so that it is not easily recognized. What operations can be applied to an existing puzzle so that the resulting puzzle looks different? (Hint: one very easy idea is to place a 2 wherever the original puzzle had a 1 and vice-versa.)
5. If there are 270725 ways to choose four cards from a deck of 52, how many ways are there to choose 48 cards from a deck of 52?
6. A circle is inscribed in an isosceles trapezoid as in Figure 3. (An isosceles trapezoid is a trapezoid where the two non-parallel sides have equal length. In Figure 3, the trapezoid is isosceles if $AD = BC$.) If segment AB has length l and segment CD has length L , how long are the other two sides, BC and DA ?

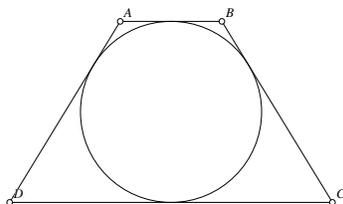


Figure 3: Isosceles Trapezoid

7. Given the following system of two equations and two unknowns, where the numbers a, b, c, d, e and f are constant:

$$ax + by = c$$

$$dx + ey = f$$

Suppose an oracle tells you that for any (well, *almost any*¹) set of values for a, b, c, d, e and f that the solution for x is given by:

$$x = \frac{ce - bf}{ae - bd}.$$

How can you find the value of y , with minimal effort?

8. What is the relationship between the following pairs of graphs: $y = x^2$ and $x = y^2$? How about $xy^3 - 3x^2y^2 = 0$ and $yx^3 - 3y^2x^2 = 0$?
9. What sorts of symmetries can you find in the graphs of the following equations. For example, which ones will be symmetric about the x -axis, the y -axis, et cetera. What other symmetries can you find?
- $y = x^2$.
 - $y = x^3$.
 - $y = x^n$, where n is a positive integer.
 - $x^2 + y^2 = 25$.
 - $x^2 + 3y^2 = 25$.
 - $y = 1/x$.
 - $x^2 - y^2 = 1$.
10. What can you say about the graph of a function f that satisfies the following conditions:
- What if $f(x) = f(-x)$, for all x ?
 - What if $f(x) = -f(x)$, for all x ?

¹To be precise, for any values such that $ae - bd \neq 0$.

- What if $f(x) = f(x + 2)$, for all x ?
 - If f is *any* function, what can be said about the graph of the function $f(x^2)$?
11. There are originally two piles of coins on a table, each of which originally contains 10 coins. A game is played by two people who alternately select a pile and remove some number of coins from that pile. The player who removes the last coin from the table wins. Does the first or second player have a winning strategy?
 12. Consider the following game. Begin with an empty circular (or rectangular) table. Players alternate moves, and it is your turn to move, you must place a quarter flat on the table. If there is no space left to do so, you lose. Does the first or second player have a winning strategy?
 13. Two players take turns placing bishops on a standard 8×8 chessboard, but once a bishop is placed, it is not moved, and no bishop can be placed on a square which is attacked by a bishop already placed. The first person who is unable to place a bishop on the board loses. Which player has a winning strategy?
 14. If you flip a fair coin 123 times, at the end are you more likely to have more heads or more tails?
 15. An urn contains 500 red balls and 400 blue balls. Without looking at them, 257 balls are removed from the urn and discarded. Finally, a single ball is drawn from the urn. What is the probability that it is red?
 16. Find the area under the curve $\cos^2 x$ from $x = 0$ to $x = \pi/2$.
 17. You have a cup of coffee and an identical cup of cream. Both contain the same amount of liquid. You take a tablespoon of cream and put it in the coffee. It is then mixed thoroughly and a tablespoon of the resulting mixture is added back to the cream. Is there now more cream in the coffee or more coffee in the cream? What if you don't mix thoroughly before you return the tablespoon of mixture to the coffee cup?
 18. A farmer with a bucket needs to water his horse. Both are on the same side of a canal that runs in a straight line. The farmer and his horse are on the same side of the canal, but the farmer needs to go to the river first to fill the bucket before he takes it to his horse. At what point on the canal should he collect the water to minimize the total distance he travels?
 19. Suppose your cue ball on a normal rectangular billiard table is at point P and the target ball is at point Q . Is it possible to hit the target after bouncing off one cushion? Two cushions? Three? How can you figure out which direction to hit the cue ball to achieve these results. (Assume that the cue ball does a "perfect" bounce each time, with the angle of incidence equal to the angle of reflection. Also assume that the table dimensions are exactly 2 : 1.)

20. In a room with rectangular walls, floor and ceiling, if a spider is on one of the surfaces and the fly on another, what is the shortest path the spider can take to arrive at the fly, if the fly does not move? (The answer, of course, will depend on the dimensions of the room, and upon where the spider and the fly initially start. What we're searching for is a method to find the solution for any room size and any initial positions of the spider and the fly.)
21. (*) If you build an elliptical pool table and you strike a ball so that it passes through one of the ellipse's foci, then after it bounces off a cushion, it will pass through the other focus. Show that this is true, based (loosely) on what you learned from the farmer and his horse a couple of problems ago. Remember that an ellipse is defined to be the set of all points such that the sum of their distances to the two foci is constant. Hint: what would the shape of a river be so that it doesn't matter where the farmer goes to get his water?
22. (*) Fagnano's problem. Show that in any acute-angled triangle, the triangle of smallest perimeter that can be inscribed in it is the so-called "pedal triangle" whose vertices are at the feet of the altitudes of the given triangle. In Figure 4, $\triangle DEF$ is the pedal triangle for $\triangle ABC$. What happens if the given triangle contains a right angle or an obtuse angle?

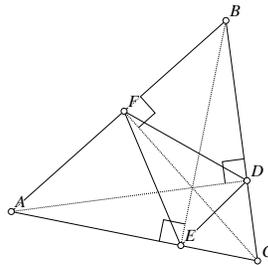


Figure 4: Fagnano's Problem

23. Fifteen pennies are placed in a triangular shape as shown in Figure 5. Many sets of three centers of those pennies form the vertices of equilateral triangles, two samples of which are illustrated in the figure. Is it possible to arrange the pennies in such a manner that no set of penny centers that form an equilateral triangle are all heads or all tails?
24. Add the whole numbers from 1 to 100.
25. Find the value of $x > 0$ which minimizes the function $f(x) = x + 1/x$.
26. What is the area of the largest rectangle that can be inscribed in a circle of radius 1?
27. In a triangle with sides 1, 1 and x , find the value of x that maximizes the area.

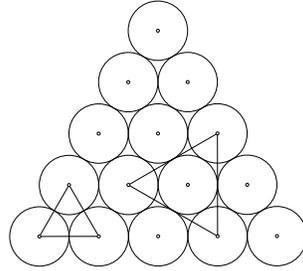


Figure 5: Fagnano's Problem

28. Give some strong evidence that an equilateral triangle is the triangle of largest area that can be inscribed in a circle. What is the largest quadrilateral that can be so inscribed? The largest n -sided figure?
29. If you have a million points inside a circle, can you find a line that divides them such that there are exactly half on each side?
30. Find all integer values a , b and c such that $a + b + c = abc$.
31. A cube is built with wire edges as in Figure 6. If wires are connected to opposite corners of the cube and a one-ampere current is passed through, how much current flows through each of the edges. (Not every edge will have the same current passing through it.)

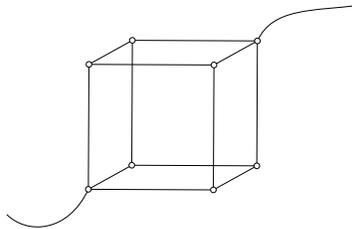


Figure 6: Wire cube

32. (*) A square metal plate has three sides held at a temperature of 100 degrees and the fourth at zero degrees. What's the temperature at the point in the center of the plate²?

²This and the following problem depend on a little bit of physics. Solutions to the heat equation (and to electrical circuits in the next problem) often satisfy the condition of superposition, meaning that a set of solutions can be added together to make the final solution. Both of these problems are of that sort.

33. (*) An infinite square mesh of wire (a small part of which is shown in Figure 7) extends in every direction. All grid lengths are equal, and all the wire has the same resistance per unit length. Two wires are connected to adjacent grid points A and B and a one-ampere current enters through A and leaves through B . What is the current through AB ? Note that the electrons will follow many paths, with more following the shorter paths since the resistance is smaller.

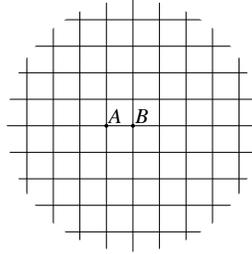


Figure 7: Infinite wire mesh

34. Evaluate the following three expressions using a (translation) symmetry observation:

$$\sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}$$

$$\frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}}$$

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

35. Solve for x :

$$2 = x^{x^{x^{\dots}}}$$

- (*) Solve for x :

$$4 = x^{x^{x^{\dots}}}$$

What is going on here?

36. How quickly can you expand the following product?

$$(x + y)(y + z)(z + x)?$$

What is different about the product?

$$(x - y)(y - x)(z - x)?$$

37. If $\{x = 1, y = 2, z = 3\}$ is a solution for the following set of equations, find five more solutions. (*) Find all solutions for x, y and z in the equations below:

$$\begin{aligned}x + y + z &= 6 \\x^2 + y^2 + z^2 &= 14 \\xyz &= 6\end{aligned}$$

38. (**) Find all solutions for w, x, y and z in the following system of equations:

$$\begin{aligned}w + x + y + z &= 10 \\w^2 + x^2 + y^2 + z^2 &= 30 \\w^3 + x^3 + y^3 + z^3 &= 100 \\wxyz &= 24\end{aligned}$$

5 Solutions to Various Problems

1. Find the sum $1 + 2 + \cdots + n$, where n is an arbitrary positive integer.

Two triangular sets from 1 to n are constructed which, when put together, form a rectangle that's $n \times (n + 1)$. Thus twice the total is $n(n + 1)$, so:

$$1 + 2 + \cdots + n = \frac{n(n + 1)}{2}.$$

2. Find the sum $1 + 3 + 5 + \cdots + 1001$. In other words, sum the odd numbers from 1 to 1001.

In the table below, think of each of the digits not as a number, but as an object. Note that there is one 1, three 3's, five 5's, et cetera. So in a sense, in the figure below, the number of digits represents the sum $1 + 3 + 5 + 7 + 9$. We can see that this makes a square containing 25 digits, so the sum above must be 25. Is it clear that each new odd number of digits can be added in the same L-shaped pattern making a square that has a side one longer than the previous.

1	3	5	7	9
3	3	5	7	9
5	5	5	7	9
7	7	7	7	9
9	9	9	9	9

We can also see that the side of the rectangle is going to be half of $n + 1$, where n is the largest odd number added. (By ignoring the 9's in the example above, for example we see a 4×4 square, and 4 is half of $7 + 1$, et cetera. Thus to add all the odd numbers from 1 to 1001, we'll obtain a square of items with $1002/2 = 501$ on each side. Thus:

$$1 + 3 + 5 + \cdots + 1001 = 501^2.$$

This problem can also be solved in exactly the same way as we summed the first two examples in this section, assuming that you are careful about counting the number of dots in the length and width of the resulting rectangle.

3. Find a general formula for the sum $1 + 3 + 5 + \cdots + (2n + 1)$.

Using exactly the same argument as above, we'll have a square with a side of $(2n + 1 + 1)/2 = (2n + 2)/2 = n + 1$, so the sum is:

$$1 + 3 + 5 + \cdots + (2n + 1) = (n + 1)^2.$$

4. Try to draw a diagram with dots that demonstrates that:

$$(n + 1)^3 = n^3 + 3n^2 + 3n + 1.$$

Hint: one nice solution is three-dimensional. What are the corresponding diagrams in two and one dimension?

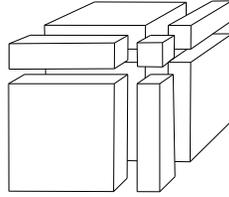


Figure 8: $(n + 1)^3 = n^3 + 3n^2 + 3n + 1$

See Figure 8. Imagine that a large cube with all three sides equal in length to $n + 1$ is sliced as in the figure, where the thin slices have width 1. The original cube has volume $(n + 1)^3$, and it is cut into eight pieces. One is a cube of side n with volume n^3 , three are plates having volume $n \cdot n \cdot 1 = n^2$ (since the thickness is 1), three are rods with volume $n \cdot 1 \cdot 1 = n$, and there is a single small cube of volume $1^3 = 1$. Adding the eight volumes together shows us that the total volume is:

$$(n + 1)^3 = n^3 + 3n^2 + 3n + 1.$$

5. How many

- corners (vertices) does a cube have? Answer: 8
- faces does a cube have? Answer: 6
- edges does a cube have? Answer: 12
- Same questions: how many vertices (V), faces (F), edges (E) has a tetrahedron?
Answer: $V = 4, F = 4, E = 6$.
- ... has an octahedron?
Answer: $V = 6, F = 8, E = 12$.
- ... has an Egyptian pyramid?
Answer (counting the base): $V = 5, F = 5, E = 8$.
- ... has a cube with a corner chopped off?
Answer: $V = 10, F = 7, E = 15$.
- ... have some other shapes?
Obviously, the answers depend on the solid, but assuming the solid has no holes, the answer has to satisfy: $V - E + F = 2$. This is known as Euler's Theorem.

6. Describe:

- intersections of a plane with a sphere
If the plane touches the sphere at a point, then there is a one-point intersection. Imagine touching a flat plate to a ball.

If the plane cuts the sphere in more than one point, it must cut a circle. The largest circle it can cut will occur when the plane passes through the center of the sphere. If this occurs, the circle of intersection is called a “great circle”. To help visualize this, think of the sphere as the earth, and look at the lines of constant latitude (constant distance north or south of the equator). All are circles and if you look at a globe of the earth, and look straight down on the north pole, all the constant-latitude lines are circles. Imagine a plane perpendicular to the axis of the north-south pole passing through the earth. At every stage it will cut a circle of constant latitude.

A few great circles on the earth are the equator or the lines of constant longitude, but there are many others. If you are forced to stay on the surface of the earth (in a boat, for example) the shortest path is along a great circle. To find that great circle, consider the origin and destination as two points and find the plane passing through those two points and the center of the earth. The path will be along the great circle which is the intersection of that plane and the surface of the earth.

- intersections of two spheres

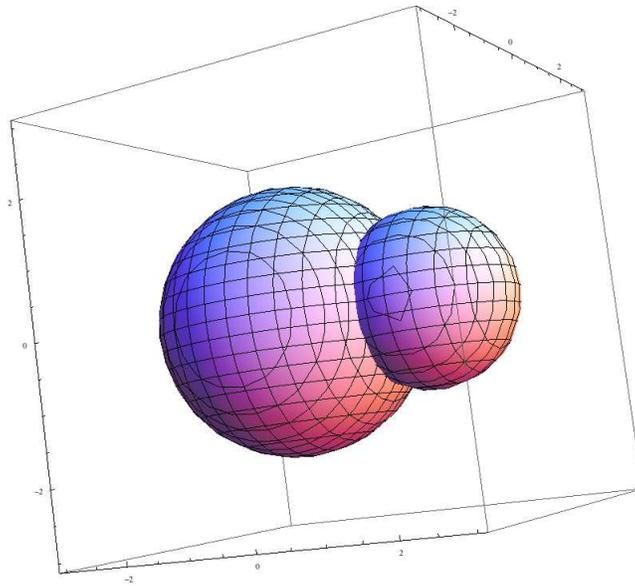


Figure 9: Two intersecting spheres

See Figure 9. As with the plane and the sphere, two spheres can touch at a point. If the intersection is larger than a point, it will be a circle (or, if the two spheres are identical, it will be the entire sphere). To see why it is a circle, imagine a line connecting the centers of the spheres, and slide a plane along that line, perpendicular to it, until it passes through a common point of the two spheres. The line will cut a circle on both spheres of

identical size, so it must be the same circle on both of them.

- intersections of a cube with a plane

This is actually a *very* difficult visualization exercise. If the plane just touches a vertex, you can get just that point. If the plane intersects an edge, you can get a line segment the length of the edge. If the plane cuts into the cube, you can get all sorts of odd-shaped polygons from triangles to hexagons. In fact, it's possible to get a perfect hexagon.

If the plane just cuts the tip of a vertex, you can get a triangle. A cut near the vertex perpendicular to the axis connecting that vertex to the opposite one on the cube will yield an equilateral triangle. If the plane is tilted, many more triangles can be obtained, ranging from equilateral to very long, skinny ones.

If you cut parallel to a face, you'll obtain a perfect square. If you imagine hanging the cube by a vertex and cutting half-way between the top and bottom vertices perpendicular to that vertex-vertex axis, that will make a perfect hexagon. By tilting away from that axis, you'll get pentagons.

7. When we draw a graph of an equation like $y = 3x + 2$, *exactly* what does the graph mean? We often look at graphs of quadratic equations in particular for points where the curve (parabola, for quadratic equations) crosses the x -axis. What does this mean?

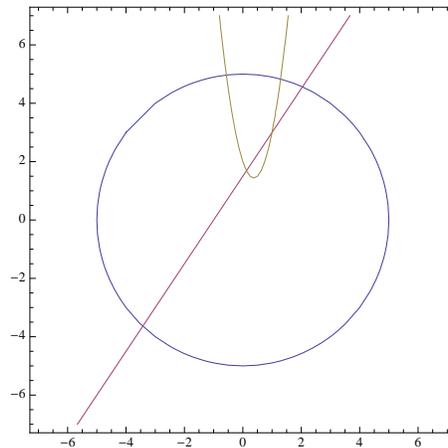


Figure 10: Graph of $(x^2 + y^2 - 25)(3x - 2y + 3)(4x^2 - 3x + 2 - y) = 0$

8. See Figure 10. We are trying to find the places where the product of three expressions is zero. If a product of numbers is zero, then one or more of those numbers must be zero. So we will have the product equal to zero if any of the following

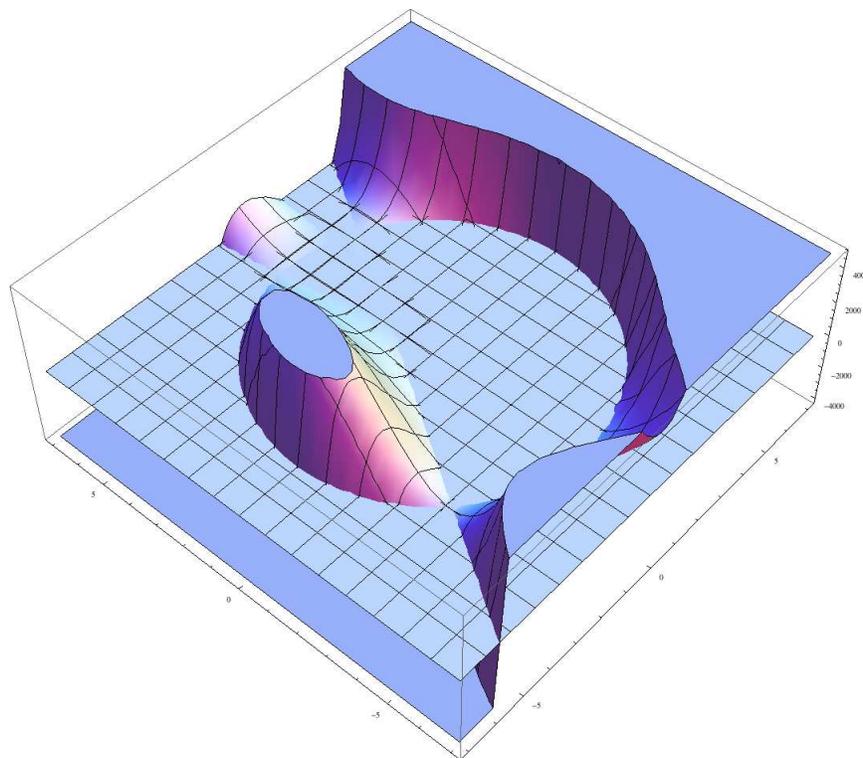


Figure 11: Graph of $(x^2 + y^2 - 25)(3x - 2y + 3)(4x^2 - 3x + 2 - y) = 0$

three equations is true:

$$\begin{aligned}x^2 + y^2 - 25 &= 0 \\3x - 2y + 3 &= 0 \\4x^2 - 3x + 2 - y &= 0\end{aligned}$$

The first is just the equation of a circle of radius 5 centered at the origin; the second is the equation of a straight line, and the third is the equation of a parabola. The figure shows all the solutions for all three.

Another way to visualize this is to imagine the surface of a three-dimensional plot of:

$$z = (x^2 + y^2 - 25)(3x - 2y + 3)(4x^2 - 3x + 2 - y).$$

This will be a complicated three-dimensional plot, but what we are interested in is the intersection of this plot with the plane $z = 0$. This is illustrated in Figure 11.

9. See Figure 12. Begin with the small square of area 1 near the upper left corner of the figure. Then rectangles and squares having areas 2, 4, 8, and so on, are added

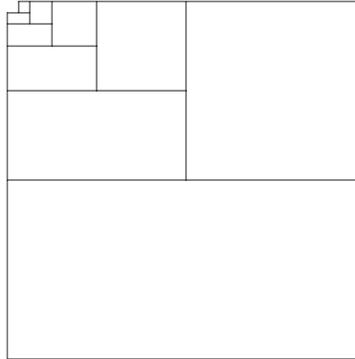


Figure 12: Adding $1 + 2 + 4 + 8 + \dots$

alternately below and to the right of the original square. Thus, for example, the sum $1 + 2 + 4 + 8$ is represented by a square in the upper right that is missing a single tiny square at its upper left corner.

Assuming the little missing square were actually there, each rectangle or square added doubles the previous area, so if there are n terms in: $1 + 2 + 4 + \dots + 2^{n-1}$, the sum of those terms will be $2^n - 1$. The “ -1 ” subtracts off the area of the upper left corner that is missing.

From these observations, we can see that:

$$1 + 2 + 4 + 8 + \dots + 128 = 256 - 1 = 255,$$

and

$$1 + 2 + 4 + 8 + \dots + 2^n = 2^{n+1} - 1.$$

10. This is based on the solution in Item 9. Note that every term in this series is exactly 3 times as big as the term in the series:

$$1 + 2 + 4 + 8 + \dots,$$

so the sum will be 3 times as large:

$$3 + 6 + 12 + 24 + \dots + 3 \cdot 2^n = 3(2^{n+1} - 1).$$

11. See Figure 13. In the figure, imagine that the area of the entire square is 1. It is successively divided, first into halves, then one of the halves is divided in half making two quarters, one of the quarters is split in half making two eighths, et cetera. At each stage of the calculation:

$$\begin{aligned} &1/2 \\ &1/2 + 1/4 \\ &1/2 + 1/4 + 1/8 \\ &1/2 + 1/4 + 1/8 + 1/16 \\ &\dots \end{aligned}$$

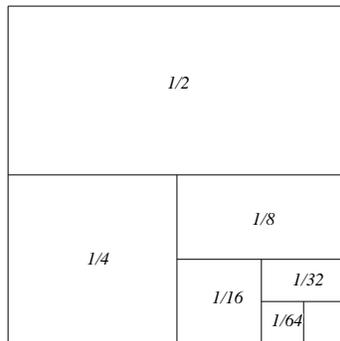


Figure 13: Adding $1/2 + 1/4 + 1/8 + \dots$

if you look at a set of areas that corresponds to that, the remainder to complete the entire square is simply a copy of the smallest rectangle/square that you obtained. Thus:

$$1/2 + 1/4 + 1/8 + \dots + 1/256 = 1 - 1/256 = 255/256.$$

12. See Item 11 and Figure 13. Each additional term in the series:

$$1/2 + 1/4 + 1/8 + 1/16 + 1/32 + \dots$$

fits inside the large square of area one, but basically cuts the uncovered part in half. Thus, as more and more terms are added, the uncovered part gets as tiny as you want, so the area gets closer and closer to 1, so it is reasonable to set the infinite sum to 1. This idea can be made rigorous mathematically (the theory of limits), but this paper is more concerned with visualization, so we will not do so here.

13. (Telescoping series) Find the finite and the infinite sum below (each term has the form $1/(n(n+1))$):

$$1/6 + 1/12 + 1/20 + 1/30 + \dots + 1/2550,$$

$$1/6 + 1/12 + 1/20 + 1/30 + \dots$$

The key here is to note that:

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Thus $1/6 = 1/2 - 1/3$, $1/12 = 1/3 - 1/4$, $1/20 = 1/4 - 1/5$ and so on. With this observation, the first sum is equal to this:

$$(1/2 - 1/3) + (1/3 - 1/4) + (1/4 - 1/5) + \dots + (1/50 - 1/51).$$

If we regroup, the sum “telescopes”, and is equal to $1/2 - 1/51 = 49/102$.

In the case of the infinite sum, the terms cancel all the way down, so the answer is just $1/2$.

14. Hint: Here is a very interesting way to write:

$$S = 1^p + 2^p + 3^p + \cdots + n^p.$$

$$S = n(1^p - 0^p) + (n-1)(2^p - 1^p) + (n-2)(3^p - 2^p) + \cdots + (1)(n^p - (n-1)^p).$$

To see this, just expand the terms and rearrange:

$$S = n \cdot 1^p - (n-1)1^p + (n-1)2^p - (n-2)2^p + \cdots + 2(n-1)^p - (n-1)^p + n^p.$$

The terms above almost cancel, and after cancellation, we obtain the correct result for n .