Rational Tangles

Or Fractions & Knots

by Kelley Barnes

Overview

What do adding positive and negative fractions have to do with tying knots? In this entertaining lesson, students will use ropes to explore and identify mathematical operations that untangle knots and lead to new thinking. Simple operations of twists and rotations circle back to practicing the addition of positive and negative fractions.

This middle-school math lesson addresses all eight Common Core Standards for Mathematical Practice, particularly, MP4: Model with mathematics and MP6: Attend to precision. More specifically, CCSS.Math.Content.7.NS.A.1d: Apply properties of operations as strategies to add and subtract rational numbers.

To ensure a successful experience, students should have the number sense to comfortably add positive numbers to negative fractions before participating in this lesson.

Objectives

While engaged in the entertaining activity of tying knots with colorful ropes, the students are unconsciously practicing their rational arithmetic. They will apply mathematical operations, on paper, to the tasks being performed, take organized notes, and be able to effectively communicate and reproduce their results. There are opportunities to volunteer and participate in front of the class without the pressure of having to do the mathematical operations on the spot. Extensions are available for those who want to explore deeper.

Required Materials

- Two ropes approximately 10 feet long and preferably two different colors. The ropes need to be thick enough that they can be easily untied. Different colors are used so the tangle structure is easier to see.
- One plastic bag or handkerchief
- Notebooks and pencils

Additional Suggested Materials

It would be helpful to make smaller tangle manipulatives in advance. Stiff paper plates and one foot long cords can be used. The cord needs to be thick enough that the knots won't become too tight to untangle. Each small group of students will need a paper plate with four slots cut equidistant along edge. Each plate needs two cords so the class can practice the tangles in their small group of 2 to 4 students. The cords are placed in the slots to mimic the actions of making tangles and performing the moves to untangle them.

Instructional Plan

It may be beneficial for some classes to start out by knowing that this lesson is about fractions. More sophisticated classes can skip this step. We will introduce the lesson by dissecting the title: ***Rational Tangles***.

What are *tangles*? What picture do you see in your mind when you hear the word *tangle*? A short discussion should lead to the idea of a "knot." Now look at the word *rational.* What does rational mean in the math context? Does anyone remember learning about *Rational Numbers*? A refresher on Rational Numbers should lead to the understanding that *rational* relates to fractions. We can write ***Fractions & Knots*** under the title ***Rational Tangles,*** where the students can easily see it.

Now for the fun part…

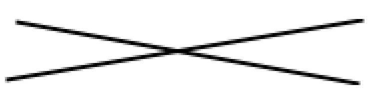
Four student volunteers are needed to demonstrate the necessary moves. Each student will hold one end of a rope. Imagine a square, or rectangle, where the students are standing at the points A, B, C, and D. Students A and B are holding opposite ends of the same rope, and are standing parallel to the wall. The student at C is standing next to B and the student at D is next to A. Students at C and D are holding opposite ends of the other rope, parallel to A and B, and are standing one step closer to the rest of the class. It is important to remember A, B, C and D refer to *the positions around this imaginary square, not the students.* Students don't keep their letters as they perform the various moves. The students should be cautioned to not tug on the ropes, or try to tighten the tangle. It's a good idea to have the volunteers wrap the rope once around their hand.

Gather four volunteers and tell them the rules about holding the rope firmly and not tugging. The first move is called a **Twist.** When **Twist** is called, the student standing in B position raises the rope and steps towards the class so the student in C position can duck under the rope and step towards the wall. The rope is then lowered back to the neutral position. The volunteers at B and C have now exchanged positions and the ropes have a "twist" in them.

A B
D C

Position B raises the rope on a **Twist**, while
Position C ducks under the rope on a **Twist**

A B
D C



After the **Twist** is performed

The second move is **Display.** When **Display** is called, A and B hold up their ropes while C and D hold down their ropes. This allows the rest of the class to see the tangle. The class should applaud this display. Practice **Twist, Twist** and **Display**.

Now for the bad news. There is no move called Untwist. We can *Twist*, but we can never *Untwist*. Instead we have another move called **Rotate.** When **Rotate** is called, volunteers move one position to the left; that is, clockwise if looking at the square from above (A moves to position B, B moves to C, C moves to D, D moves to A). Students perform the **Rotate**. Now **Display**. Does this tangle look different than the last tangle? Do you remember what the last tangle looked like? We have a very easy way to show it to you. Please **Rotate**, **Display.** Now the volunteers have changed positions but this is exactly the same tangle we made after three **Twists**. Let's practice a few more moves: **Rotate, Twist, Rotate, Display**. The volunteers can return to their seats. Thanks for your help.

Now give the students a minute or two to talk to each other and formulate questions. While they are talking, untangle the ropes and return them to their parallel starting position. Ask for any questions, and record them on the board. Hopefully, “What does this have to do with Math?” will be one of the questions. “Can we always untangle a Knot?” is another excellent question.

Explain that it will help to have a shorthand notation to record our actions. We will use **T** and **R** to represent **Twist** and **Rotate**, respectively. We don’t need a shorthand notation for **Display** because it doesn’t change the tangle. We will sometimes use exponents to represent multiple, consecutive moves. For example: $\mathbf{T^2}$ means the same thing as **TT:** both represent two consecutive **Twists**.

Where does the Math come in?

Now we need four more volunteers. Remind them about the rules for not letting go or tugging on the rope. Explain that the starting position, two parallel ropes, denotes zero position. It makes no difference which rope is in front. When the ropes are parallel to the wall, they are in the zero position. The only reason that the ropes are different colors is to make the tangle structure easier to see.

Students please **Twist**. We are going to give **Twist** the mathematical property of adding 1. Therefore, starting at zero and adding one **Twist**, we have:

$$0 + T \Rightarrow 0 + 1 \Rightarrow 1 \text{ which will be written as } 0 \ ^{T}\ 1$$

> Side Note: In equations, when a Twist is applied it will be written as ^{T} and the operation this refers to is *add 1* or +1. Two Twists are written as ^{TT} and the resulting operation is +1+1 or +2. A Rotation will be written as ^{R} and we will eventually determine that this operation is turning a number into its negative reciprocal.

Let’s try to undo this **Twist**. Remember we don’t have a move called Untwist, so let’s try **Rotate**. Please **Rotate** and **Display**. Sadly, we haven’t untangled it yet. We determined earlier that two **Rotates** in a row is unhelpful because it brings us back to the same tangle, so let’s do another **Twist** instead. Please **Twist** and **Display**. And we are back to the zero position! The volunteers can be thanked and dismissed.

Let’s look at what we did: shorthand **TRT** (remember the students haven’t discovered what operation ^{R} does so we can’t fill in the number after ^{R} yet)

$$0 \ ^{T}\ 1 \ ^{R}\ \underline{\quad}\ ^{T}\ 0$$

We know that **Twist** adds 1 so the blank must contain -1 (so that ^{T} will convert -1 back to zero)

$$0 \ ^{T}\ 1 \ ^{R}\ \underline{-1}\ ^{T}\ 0$$

So now the question is: What is the mathematical operation of **R** that causes a 1 to become a -1? Give the students a minute or two to come up with ideas. Record ideas on the board. Depending on the sophistication of the students and the time constraints, you may need to help them come up with ideas.

There are two obvious (but incorrect!) hypotheses: **R =>** *subtract 2*; or **R** => *multiply by -1*.
The correct operation for some number x is: **R** => -1/x. Feel free to provide this option if it isn't suggested.

Point out the title of this Math Lesson with its reinterpretation and ask the students to vote on which of these ideas they think is the correct operation for **Rotate**.

How do we figure out which of these excellent ideas is the actual solution? Let's practice our moves and come up with some ways to untangle relatively easy tangles such as $\mathbf{T^2}$ and $\mathbf{T^3}$. At this point each team of students can be given a personal size, tangle board manipulative. Explain that they need to be systematic in their use of the board. To perform a **Twist**, they swap the cords on always the same side of the board and to perform a **Rotate**, they simply rotate the whole board a quarter turn clockwise.

Their objective is to untangle $\mathbf{T^2}$ and $\mathbf{T^3}$, *and be able to communicate and reproduce their results.* Advise them to take notes in the established shorthand. It may be helpful to designate one member of the group to be the scribe and take careful notes. Once they think that they have a possible solution, they should be able to reproduce their results. Students who finish quickly can try $\mathbf{T^4}$ or $\mathbf{T^5}$. Did they discover any patterns which will allow them to easily untangle $\mathbf{T^6}$ or $\mathbf{T^7}$ or even $\mathbf{T^n}$?

Give the students five minutes or so to work on this in their small groups, then call them back to focus. Record their possible solutions for $\mathbf{T^2}$**,** $\mathbf{T^3}$**,** etc. Now we have more data to help us figure out what operation **Rotate** performs.

For reference, here are the sequences to untangle our sample tangles as well as a general formula.

Tangle	Untangle
T	RT
TT = T^2	RT RTT
TTT = T^3	RT RTT RTT
TTTT = T^4	RT RTT RTT RTT
T^n	RT $(RTT)^{n-1}$

Let's Check Our Work

Now we are ready for four new volunteers to try these untangle sequences. Start with: **Twist, Twist,** and **Display**. Now we can test out the possible sequences that untangle $\mathbf{T^2}$. Here you can try any untangle sequences

that the students discovered on their own, ending with the one listed in the table above: **Rotate, Twist, Rotate, Twist, Twist, Display.** It works!
Let's look at the equations now:

$$0 \ ^{TT}\ 2 \ ^{R}\ \_\_ \ ^{T}\ \_\_ \ ^{R}\ \underline{-2} \ ^{TT}\ 0$$

We know there has to be a -2 before the final **TT** to bring the tangle back to 0. Let's try our possible operations for **Rotate**. Write the following sequences directly below the above sequence.

If R => *subtract 2* then: $0 \ ^{TT}\ 2 \ ^{R}\ 0 \ ^{T}\ 1 \ ^{R}\ -1 \ ^{TT}\ 1$: Doesn't work because we need to end with 0.
Here are the detailed steps: 0 + 1 + 1 => 2 – 2 => 0 + 1 => 1 – 2 => -1 + 1 + 1 => 1

If R => *times (-1)* then: $0 \ ^{TT}\ 2 \ ^{R}\ -2 \ ^{T}\ -1 \ ^{R}\ 1 \ ^{TT}\ 3$: Doesn't work!
0 + 2 => 2(-1) => -2 + 1 => -1(-1) => 1 + 2 => 3

If the R operation is *-1/x* then: $0 \ ^{TT}\ 2 \ ^{R}\ -1/2 \ ^{T}\ 1/2 \ ^{R}\ -2 \ ^{TT}\ 0$: It works!
0 + 2 => 2(-1/x) => -1/2 +1 (or +2/2) => 1/2(-1/x) => -2 + 2 => 0

Now let's do a slightly more complicated tangle and start with **3 Twists**. The volunteers can perform the actions while the math is demonstrated on the board.

Twist^3 Rotate [Display] Twist Rotate Twist^2 [Display] Rotate Twist^2

$0 \ ^{TTT}\ 3 \ ^{R}\ -1/3 \ ^{T}\ (-1/3 + 3/3) = 2/3 \ ^{R}\ -3/2 \ ^{TT}\ (-3/2 + 4/2) = ½ \ ^{R}\ -2 \ ^{TT}\ (-2 + 2) = 0$

^{TTT} => 3 ^{T} => Adding 1 ^{TT} => Adding 2 ^{TT} => Adding 2

Thank and dismiss the volunteers. Explain there are some neat patterns that appear when we look at the math. We start by Twisting several times, making a positive whole number. Rotate turns this into its negative reciprocal, which is a negative fraction. Then we Twist as many times as necessary to make a positive fraction. Rotate turns our number into its negative reciprocal, and we repeat this process until we have a whole negative number that we can Twist until zero!

If time allows, do one more example. We need another set of volunteers. This tangle can be made by **Twist, Twist, Twist, Rotate, Twist, Twist, Twist.** The resulting starting fraction will be 8/3. As a group, determine the untangle sequence, (**T^3** took 8 steps to untangle, **T^{3}RT^3** will take 12 steps to untangle). Thank and dismiss the volunteers.

The Grand Finale… the last 10 - 15 minutes

The math is pretty convincing. How many people think we can undo any tangle by using pure mathematics? Let's try it with our final four volunteers. Start with a number of **Twists** (let the students decide how many, less than 5), then a **Rotate**, then more **Twists** (less than 5), etc. ending with **Display**. While the moves are being performed, keep track of the math on the board, while the students record the equation in their workbooks, until you have a sufficiently, but not overwhelmingly, complicated fraction to untangle. Now for a cool trick! Take a plastic bag, poke two holes in the bottom and very carefully thread both of the left hand ropes through the holes in the bag, returning the ropes to the proper volunteers. Tie the opening of the bag shut around the knot to completely enclose the tangle. Work together to decide what moves the volunteers need to perform to untangle the knot.

While working through the problem on the board and in the workbooks, perform the moves to untangle the tangle, which is now hidden inside the plastic bag. After the equation reaches zero, **Display**. There will now be a horrible snarl of rope and plastic bag that makes it look like the experiment failed. Dramatically, untie, rip apart, or carefully cut the plastic bag away, give the ropes some gentle shaking or tugs and the knot will magically fall back to the zero position (provided the math was all correct)!

Extensions: Not Just Knots

This lesson can easily be extended to two sessions or a longer single session.

1. An Alternate Introduction

One way to introduce the lesson could be to have a teaser demonstration the day before. Four volunteers could be called up and taught how to perform the three basic moves. The class could make up a random tangle while the teacher keeps track of the math on a small notepad. While solving the tangle silently, the teacher then calls out the proper moves to untangle the knot back to the zero position. The demonstration is over and the students are told the lesson will be continued the next day.

2. Infinity

What happens if we start with a **Rotate**? This is best shown as a demonstration. Starting in the zero position, have the volunteers **Rotate**. This strange configuration, represented by -1/0 is a nonsense value that we could argue sort of behaves like “infinity”. If we then apply a **Twist**, nothing changes. We can apply multiple **Twists** and still nothing changes! Just like if we added 1 to infinity, we would still have infinity, $\infty + 1 = \infty$.

Interestingly, four **Rotates** bring the volunteers back to their starting positions. Some students may be concerned that after a tangle is untangled, the rope colors are sometimes switched. Two **Rotates** will solve this discrepancy.

3. Is There a Knot That Can’t Be Untangled?

Students can practice the math on their own or in small groups. They can be given starting fractions, such as 3/5 or 7/11 and asked to get the number to zero using Twists and Rotates. They can create their own tangles on paper, untangle it on paper, and then demonstrate to the class, with the ropes, that their equations work. This could be an exercise for early finishers in other activities.

Have the students prove that any fraction, no matter how ugly, can always be converted to zero using these two steps. Looking at a long tangle, you will see that the denominators progressively get smaller over time, so they must eventually get to one.